



## An Approximate Model for the Theoretical Prediction of the Velocity Increase in the Intermediate Ballistics Period

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**Abstract:** This paper is a continuation of an earlier work [1] by the authors devoted to CFD modelling of the intermediate ballistics period. A semi-analytical model is proposed based on the analytical solution of the Prandtl-Meyer expansion for an approximate assessment of the magnitude of the velocity increase in the intermediate period. Two variants of the model provide upper and lower estimates of the velocity increase. A comparison of these estimates with the results of CFD modelling gives a similar magnitude for the velocity increase. A formula is proposed for a simple estimation of the velocity increase that can be used to calculate a correction to the muzzle velocity value obtained by internal ballistics models.

**Keywords:** intermediate ballistics, internal ballistics, projectile velocity

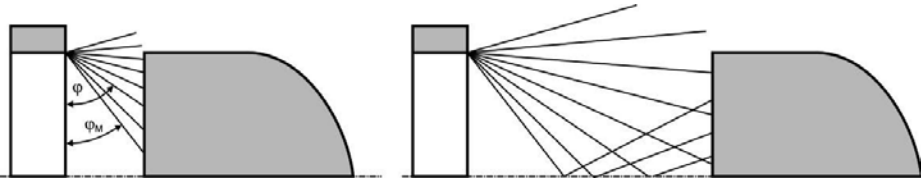
### 1 Introduction

Our previous work [1] was devoted to the presentation of the results of modelling of the intermediate ballistics using CFD methods. The aim of this modelling was to estimate the magnitude of the velocity increase  $\Delta u$  in the intermediate ballistics period. It was shown that the relative increase in velocity is generally less than 1% and only for very low muzzle velocities does it exceed this value. Although the CFD model predicts the velocity increase quite precisely, it cannot be used in the fire control systems, because it needs a long computing time. That is why a simple model, requiring only several seconds of computing time,

is desirable. In this paper we present an approximate, semi-analytical model of the intermediate ballistics. This model is based on the same physical model as that presented in [1]. However, a completely different approach to solving the mathematical model is applied. The model can be easily implemented in fire control systems. Moreover, based on the results of [1], a simple formula is proposed for estimating the value of  $\Delta u$ . The formula can be used for calculating a correction to the muzzle velocity calculated using internal ballistics models.

## 2 Description of the Model

The results of the modelling presented in [1] proved that the period of time in which the projectile undergoes acceleration after leaving the muzzle is very short. During this period the flow near the muzzle is dominated by the Prandtl-Meyer (P-M) expansion and its reflection at the axis (Figure 1).



**Figure 1.** Structure of the flow near the muzzle.

An approximate model can be proposed for estimating the mean pressure acting on the tail of a projectile based on the analytical solution for plane P-M flow. The solution is given by the following relations [2]:

$$u_\varphi = c, \quad u_R = \sqrt{2\Delta w + u_m^2 - c^2}, \quad u = \sqrt{u_R^2 + u_\varphi^2}$$

$$\varphi = \varphi_M - \int_{\rho_m c_m}^{\rho c} \frac{d(\rho c)}{\rho u_R}, \quad \Delta w = \int_p^{p_m} \frac{dp}{\rho}, \quad \varphi_M = \arccos\left(\frac{u_m}{c_m}\right) = \arccos(M_m^{-1}) \quad (1)$$

where:  $u_R, u_\varphi$  mean normal and tangential components of the velocity of flow  $u$ , respectively;  $\rho$  – density;  $p$  – pressure;  $c$  – sound velocity;  $Dw$  – specific enthalpy increment;  $u_m, p_m, r_m, c_m, M_m, \varphi_M$  – muzzle velocity, pressure, density, velocity of sound, Mach number and Mach angle, respectively.

For a perfect gas with the exponent  $k$ , the following relation between pressure  $p$  and the angle  $\varphi$  can be derived from the relations (1):

$$p = p_m \left\{ \cos \left[ w_2 (\varphi - \varphi_M) \right] - \sqrt{w_3 - 1} \sin \left[ w_2 (\varphi - \varphi_M) \right] \right\}^{w_1}$$

$$w_1 = \frac{2k}{k-1}, \quad w_2 = \sqrt{\frac{k-1}{k+1}}, \quad w_3 = \frac{2 + (k-1)M_m^2}{k+1} \quad (2)$$

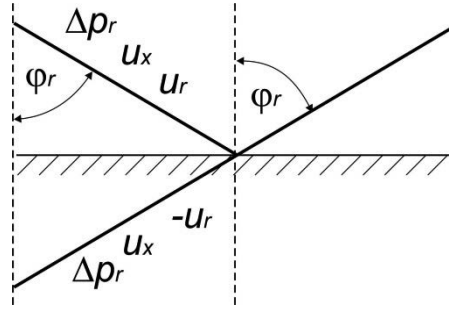
Based on formula (2), we can find the value of the static pressure at the tail of the projectile at a point reached by a characteristic with inclination angle  $\varphi$ . In order to estimate the value of the dynamic pressure, we can make use of the following formula for the axial component of the velocity of the flow:

$$u_x = c_m \left( \sqrt{\frac{2}{k-1} + M_m^2 - \frac{k+1}{k-1} s^{2w_4}} \sin \varphi + s^{w_4} \cos \varphi \right), \quad s = \frac{p}{p_m}, \quad w_4 = w_1^{-1} \quad (3)$$

An approximate value of the dynamic pressure can be calculated using the following formula:

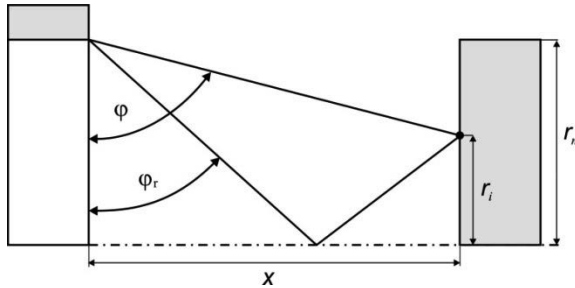
$$\Delta p = \rho c (u_x - u_m) = k p_m s^{w_5} \left( \frac{u_x}{c_m} - M_m \right), \quad w_5 = \frac{2k}{k+1} \quad (4)$$

In this formula the velocity of the projectile is assumed to be equal to  $u_m$ , which is a good approximation, taking into account the fact that the velocity increase is small. The sum of  $p$  and  $\Delta p$  gives an approximate value of the pressure at a point at the tail of the projectile, which is reached by a characteristic having an inclination angle  $\varphi$ . However, this approximation is valid only until the moment when the reflected wave reaches that point. In order to take into account the reflection, we used the acoustic approximation. For this approximation the reflection of a wave can be considered as a superposition of two waves. The waves are symmetric against the reflecting plane (we used the solution for plane symmetry). This means that, at a given characteristic, both waves have the same value of the pressure and the axial velocity component, while their radial components of velocity have equal values but opposite signs (see Figure 2). Therefore, we can compute  $p_r$  and  $\Delta p_r$  for the reflected wave using formulae (2) and (4).



**Figure 2.** Illustration of the idea of reflection of the wave at the plane of symmetry.

The procedure for the estimation of the mean pressure is as follows. For a given position of the projectile, pressure is calculated at  $n+1$  points, having radial coordinate  $r_i = r_m(i-1)/n$ . For each point, the inclination angles of the characteristics of incident and reflected waves are calculated (Figure 3 – characteristics are assumed to be rectilinear).



**Figure 3.** Illustration of the procedure for the calculation of the inclination angles.

$$\varphi = \begin{cases} \operatorname{arctg}\left(\frac{x}{r_m - r_i}\right), & i = 1, n, \\ \frac{\pi}{2}, & i = n + 1 \end{cases} \quad \varphi_r = \operatorname{arctg}\left(\frac{x}{r_m + r_i}\right), \quad i = 1, n + 1 \quad (5)$$

For a given value of  $\varphi$ , we calculate  $p$  and  $\Delta p$ . Then, for a given value of  $j_r$  we calculate  $p_r$  and  $\Delta p_r$ . If  $\varphi < \varphi_M$ , we put  $p = p_m$ ,  $\Delta p = 0$ . Similarly, if  $j_r < \varphi_M$ , we put  $p_r = p_m$ ,  $\Delta p_r = 0$ . We then calculate the resultant pressure, assuming linear interference of the waves:

$$p_i = p + p_r - p_m + \Delta p + \Delta p_r \quad (6)$$

The mean value of the pressure acting on the tail of the projectile is calculated as:

$$p_{av} = \frac{1}{2} \sum_{i=1}^n (\bar{r}_{i+1}^2 - \bar{r}_i^2) (p_i + p_{i+1}), \quad \bar{r}_i = \frac{r_i}{r_m} \quad (7)$$

Taking into account that  $x = u_m \cdot t$  (we can neglect the velocity increase in assessing the distance the projectile has moved), we can find the dependence of the mean pressure on time and then integrate the equation of motion of the projectile. The results of [1] suggest that the mean pressure value acting on the forward part of the projectile can be neglected.

The model described above does not take into account the cylindrical symmetry. Due to this symmetry, the pressure in the P-M flow decreases faster than for plane symmetry. Therefore, we can expect that the approximate model will provide an upper estimate of the pressure values.

We can take the cylindrical symmetry into account in an approximate manner. The continuity equation for cylindrical symmetry has the form:

$$\frac{\partial p}{\partial t} = -\frac{\partial(\rho u_x)}{\partial x} - \frac{\partial(\rho u_r)}{\partial r} - \frac{\rho u_r}{r} \quad (8)$$

For an isentropic flow we can represent it in the form:

$$\frac{\partial p}{\partial t} = -c^2 \left[ \frac{\partial(\rho v_x)}{\partial x} - \frac{\partial(\rho u_r)}{\partial r} \right] - \frac{k p u_r}{r} \quad (9)$$

The first sum and on the right hand side of formula (9) represents the pressure derivative for plane symmetry. Thus, integrating formula (9) along the projectile trajectory we obtain:

$$\Delta p = \Delta p_p - \int_0^l \frac{k p(u_m \tau, r, \tau) u_r(u_m \tau, r, \tau)}{r} d\tau \quad (10)$$

We can treat the integral on the right hand side of formula (10) as a correction

term to the solution for plane symmetry. Therefore, we can calculate  $\Delta p_P$  in the way described above and then correct its value. We estimate values of the radial velocity component  $u_r$  from the relation for plane symmetry:

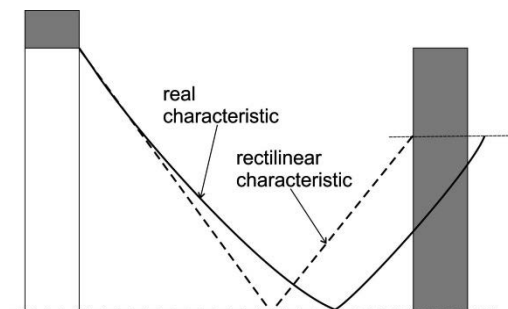
$$u_r(\varphi) = c_m \left( -\sqrt{\frac{2}{k-1} + M_m^2 - \frac{k+1}{k-1} s^{2w_4} \cos \varphi} + s^{w_4} \sin \varphi \right) \quad (11)$$

In the case of interference of the incident and reflected waves, we calculate the value of  $u_r$  as:

$$u_r = u_r(\varphi) - u_r(\varphi_r) \quad (12)$$

The integral in formula (10) is calculated numerically, making use of the values of  $p_i$  and  $u_{ri}$  in consecutive time intervals.

We can expect the above approximation to give a lower estimate of the pressure because treating the characteristics as rectilinear causes a given characteristic to reach the projectile tail sooner than it should (Figure 4). This causes the pressure to drop faster than for the real characteristics.

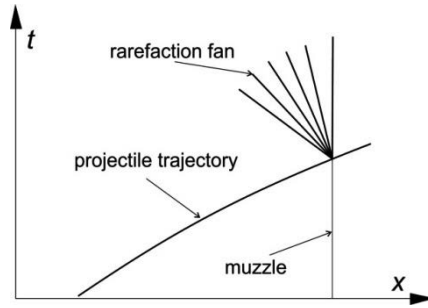


**Figure 4.** Illustration of the difference between rectilinear and real characteristics.

The approximate model can be used with some modifications in the case when the velocity of the propellant gases at the starting moment of the intermediate ballistics period is subsonic. In this case, a rarefaction wave enters the muzzle. This accelerates the gases and as a result the flow at the muzzle becomes sonic. This means that the first characteristic of the P-M flow is positioned at the muzzle. Therefore, we can use the approximate model described above to calculate the average pressure acting on the projectile. However, we need to know the new parameters at the muzzle, which are different from the results of the internal

ballistics calculations. For this reason, we propose the following approximation.

We assume that at the initial moment, a centred rarefaction fan is generated inside the barrel (Figure 5). Based on this approximation, we can calculate the velocity and pressure at the muzzle, based on the relation for the centred rarefaction wave:



**Figure 5.** Wave diagram showing the rarefaction fan generated at the moment the projectile leaves the muzzle.

$$u_0 + \frac{2}{k-1}c_0 = u_m + \frac{2}{k-1}c_m \tag{13}$$

where  $u_0$  and  $c_0$  mean flow velocity and velocity of sound at the final moment of the internal ballistics period respectively.

Because  $u_m = c_m$  at the muzzle, we can obtain the following formula from (13):

$$u_m = c_m = \frac{k-1}{k+1} \left( u_0 + \frac{2}{k-1}c_0 \right) = \frac{k-1}{k+1}c_0 \left( M_0 + \frac{2}{k-1} \right) \tag{14}$$

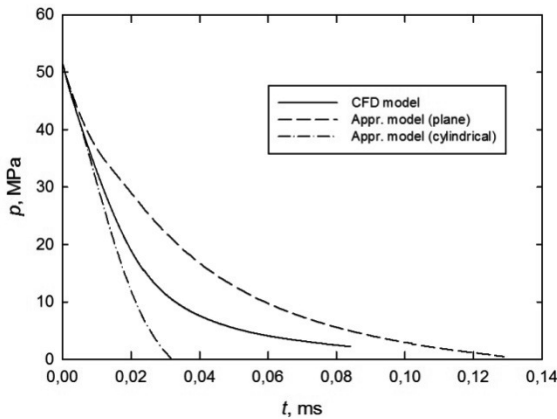
The values of the other parameters at the muzzle can be calculated by the formulae:

$$\rho_m = \rho_0 \left( \frac{c_m}{c_0} \right)^{\frac{2}{k-1}}, \quad p_m = \frac{\rho_m c_m^2}{k}, \quad M_m = 1 \tag{15}$$

We can use the calculated values in the formulae derived earlier, except for formula (4), where we substitute  $u_0$  and  $M_{m0} = u_0/c_m$  for  $u_m$  and  $M_m$ , respectively.

### 3 Results and Discussion

The results obtained with the approximate model are compared with the results of the model described in [1] (subsequently referred to as the CFD model) in Figure 6. This presents the time dependence of the calculated mean pressure acting on the projectile tail for a 30 mm caliber launching system. Zero time corresponds to the moment when the projectile leaves the muzzle and the expansion of the propellant gases begins (this also applies to Figures 7 and 8). The results from the approximate model differ considerably from the CFD model results. However, the variants of the approximate model (without and with the correction for cylindrical symmetry) give upper and lower estimates of the pressure values. This means that they also give upper and lower estimates of the velocity increase. The calculated  $\Delta u$  values for a 30 mm launching system are given in Table 1. The first row refers to standard ammunition, whilst the second and third rows refer to ammunition with 20% larger and 20% lower powder mass respectively. In all cases, the values of the velocity increase calculated by the CFD model are between the values calculated by the two variants of the approximate model and quite close to the average of the upper and lower estimates. This result strengthens our belief that the CFD model properly predicts the magnitude of the velocity increase in the intermediate ballistics period.



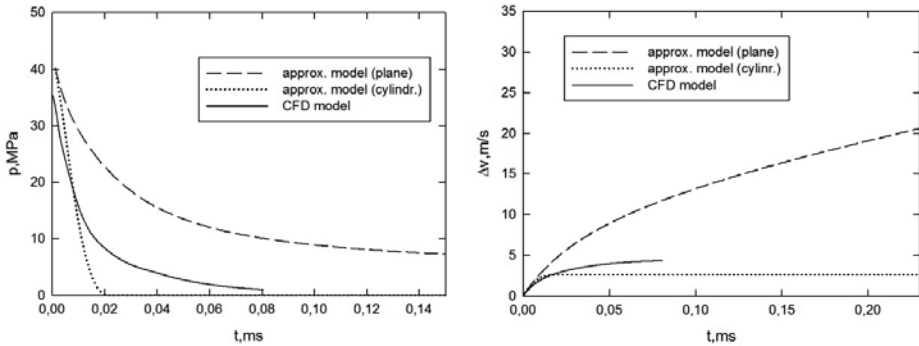
**Figure 6.**  $p(t)$  plots for the CFD model and two variants of the approximate model (30 mm launching system).



**Table 1.** Results of calculations of the velocity increase in the intermediate ballistics period by the CFD model and the two variants of the approximate model (plane and cylindrical) for a 30 mm launching system

Variant	$Dv_{CFD}$ , [m/s]	$Dv_{plane}$ , [m/s]	$Dv_{cylindr}$ , [m/s]	$Dv_{av}$ , [m/s]
standard powder mass	2.05	3.28	1.31	2.295
20% larger powder mass	1.863	2.762	1.03	1.896
20% smaller powder mass	2.128	3.385	1.25	2.318

The results obtained for a subsonic case are illustrated by the  $p(t)$  and  $u(t)$  curves, calculated for a 9 mm launching system, shown in Figure 7. In this case, the differences between the results of the approximate model and the CFD model are much larger than those for the supersonic case, especially for the variant for plane symmetry which differs greatly from the results for the CFD model. The reason for this is obvious, because in the subsonic case the propellant gases are accelerated in the radial direction much more intensely than in the supersonic case.



**Figure 7.**  $p(t)$  and  $u(t)$  plots for the CFD model and the two variants of the approximate model (9 mm launching system).

The approximate model can be used for a rough estimation of the velocity increase in the intermediate ballistics period. However, based on the results of the CFD model, a much simpler estimation can be proposed.

In our previous paper [1], we invoked results of measurements of the acceleration of projectiles presented in [3] and [4] in order to validate the

CFD model. In these papers, the results of measurements of the acceleration of projectiles in the intermediate period were used for determining the pressure acting on the projectile. The dependence of the pressure on time was approximated by an exponential decay:

$$p(t) = p_0 e^{-\beta t} \quad (16)$$

We found that such an approximation fits well to our results from the CFD modelling. Making use of this approximation, we obtained the following form of the projectile equation of motion:

$$m_p \frac{du_p}{dt} = S_p p_0 e^{-\beta t} \quad (17)$$

Integrating this equation, we obtained:

$$\Delta u_p(t) = \frac{S_p p_0}{\beta m_p} (1 - e^{-\beta t}) \quad (18)$$

where  $S_p$  means the area of the tail of the projectile and  $m_p$  its mass.

The coefficient  $\beta$  has the dimension of the reciprocal of time. Assuming that the muzzle velocity  $u_0$  is a characteristic velocity, while the caliber  $d$  is the characteristic length, we can express the coefficient  $\beta$  by as a non dimensional coefficient  $\gamma$  in the following way:

$$\beta = \gamma \frac{u_0}{d} \quad (19)$$

In general, the coefficient  $\gamma$  may be a function of several non dimensional parameters characterizing the process. However, we assume that it is only a function of the muzzle Mach number  $M_m$ . From formulae (18) and (19) we have:

$$\frac{\Delta u_{p \max}}{u_0} = \gamma(M_m) \frac{S_p p_0 d / 2}{m u_0^2 / 2} \quad (20)$$

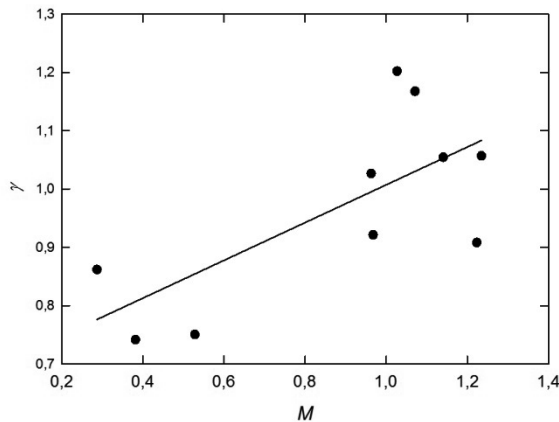
This formula suggests that the relative velocity increase is proportional to the ratio of the work done by the muzzle pressure on a path equal to half of the caliber to the kinetic energy of the projectile at the muzzle. In order to find the values of the coefficient  $\gamma(M_m)$ , the results of calculations presented in [1] for

various launching systems are used. These are presented in Table 2. Based on the  $\Delta u_{CFD}$  values, the coefficients  $\gamma$  have been estimated and the results are shown in Figure 8.

**Table 2.** Results of calculations of the velocity increase in the intermediate ballistics period by the CFD model for various launching systems and by formulae (20) and (21)

Caliber, [mm]	$\Delta u_{CFD}$ , [m/s]	$\Delta u_{pmax}$ , [m/s]	Error, [%]
5.56 (1)*	2.68	2.75	2.5
5.56 (2)**	3.40	3.00	-11.8
7.62	2.70	2.92	8.2
9	5.11	5.56	8.8
12.7	2.50	2.85	13.8
14.5	5.96	5.04	-15.5
23	3.80	3.68	-3.1
30	2.38	2.83	18.9
122	2.03	1.83	-9.9
155	5.01	5.00	-0.2

\*Assault rifle Beryl; \*\* New assault rifle under development.



**Figure 8.** Dependence of the coefficient  $\gamma$  on the Mach number  $M_m$ .

The dependence of  $\gamma$  on the Mach number  $M_m$  is approximated by the linear relation:

$$\gamma(M_m) = 0.74 + 0.24M_m \quad (20)$$

The assumption that the coefficient  $\gamma$  is only a function of the Mach number  $M_m$  is a rough approximation. This is demonstrated by the differences between the regression line in Figure 8 and the points corresponding to the calculated values of  $\gamma$ . Nevertheless the calculated values of the velocity increase differ from the results from the CFD model by no more than 20% (see Table 2). This precision is acceptable because the relative increase in the velocity, as a rule, does not exceed 1% of the muzzle velocity. Therefore, the proposed formulae can be used to calculate a correction to the muzzle velocity determined by the internal ballistics models, which provide values of  $u_0$ ,  $p_0$  and  $M_0$ .

## 4 Conclusions

The main results of the work can be summarized as follows:

1. The estimates obtained for the projectile velocity increase in the intermediate ballistics period give an upper and a lower limit of the results obtained by the use of the CFD model described in [1].
2. The average value of the two estimates gives a good approximation of the results obtained by the CFD modelling for the case when the flow of propellant gases at the muzzle is supersonic. In the case of subsonic flow, only the lower estimate can be used as an approximate value of the velocity increase.
3. The proposed approximate formula approximates, with acceptable accuracy, the results of the CFD modelling for various launching systems.

## 5 References

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